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Symmetric positive solutions of fourth order boundary value problems for an increasing homeomorphism and homomorphism on time-scales

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ABSTRACT

Let $\mathbb{T} \subset \mathbb{R}$ be a symmetric bounded time-scale, with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$. We consider the following fourth order boundary value problem

$$\phi(-px^{\Delta\nabla})^{\Delta\nabla}(t) + f(t, x(t)) = 0, \quad t \in \mathbb{T}_{\kappa^2}^{\kappa^2},$$
$$x(a) = x(b) = 0, \qquad x^{\Delta\nabla}(\sigma(a)) = x^{\Delta\nabla}(\rho(b)) = 0$$

for a suitable function p and an increasing homeomorphism and homomorphism ϕ. By using the Krasnosel'skii fixed point theorem, we present sufficient conditions for the existence of at least one or two symmetric positive solutions of the above problem on time-scales. As applications, two examples are given to illustrate the main results.

1. Introduction

The theory of dynamic equations on times-scales was introduced by Stefan Hilger in this Ph.D. thesis in 1988 [1]. The time-scales approach, not only unifies differential and difference equations, but also provides accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. By using the theory of time-scales we can also study biological, heat transfer, economic, stock market and epidemic models [2–5]. Hence, the study of dynamic equations on time-scales is worthwhile and has theoretical and practical values [6–8]. In the past few years, it is found that a considerable amount of interest and research in this area is rapidly growing.

In this paper, we are concerned with the existence of symmetric positive solutions of the following fourth order boundary value problem (FBVP)

$$\phi(-px^{\Delta\nabla})^{\Delta\nabla}(t) + f(t, x(t)) = 0, \quad t \in \mathbb{T}_{\kappa^2}^{\kappa^2}, \tag{1.1}$$

$$x(a) = x(b) = 0, \qquad x^{\Delta\nabla}(\sigma(a)) = x^{\Delta\nabla}(\rho(b)) = 0 \tag{1.2}$$

where $\mathbb{T}$ is a symmetric time scale, i.e., $b - t + a \in \mathbb{T}$ for any given $t \in \mathbb{T}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and homomorphism with $\phi(0) = 0$. A projection $\phi : \mathbb{R} \to \mathbb{R}$, which generates the p-Laplacian operator $\phi_p(u) = |u|^{p-2}u$ for $p > 1$, is called an increasing homeomorphism and homomorphism if the following conditions are satisfied.

(i) If $x \le y$, then $\phi(x) \le \phi(y)$, for all $x, y \in \mathbb{R}$.
(ii) ϕ is a continuous bijection and its inverse mapping is also continuous.
(iii) $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in \mathbb{R}$.

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Recently, for the existence problems of positive solutions of boundary value problems on time-scales, some authors have obtained many results; for details, see [9–14] and the references therein. However they did not further provide characteristic of positive solutions, such as symmetry that not only has its theoretical value, such as in studying chemical structures [15].

Motivated by the Refs. [16,17], we consider the FBVP for an increasing homeomorphism and homomorphism (1.1)–(1.2) on symmetric time scales. By using the symmetric technique and the Krasnosel'skii fixed point theorem, we obtain the existence of one or two symmetric positive solutions of problem (1.1)–(1.2). As applications, two examples are given to illustrate our main results. These results are new for the special cases of continuous and discrete equations, as well as in the symmetric time-scale.

The rest of the paper is organized as follows. In this section, we give some definitions and lemmas. In Section 2, by using the Krasnosel'skii fixed point theorem, we obtain the existence of one or two symmetric positive solutions of the problem (1.1)–(1.2) and also we present two examples to illustrate our main results.

We first briefly recall some basic definitions and results concerning time-scales. Further general details can be found in [6–8].

Let $\mathbb{T} \subset \mathbb{R}$ be a bounded time-scale (a non-empty closed subset of $\mathbb{R}$), with $a = \min\{s \in \mathbb{T}\}$, $b = \max\{s \in \mathbb{T}\}$. Define the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ where, in this definition, we write $\inf \emptyset = a$, $\sup \emptyset = b$ so that $\rho(a) = a$, $\sigma(b) = b$. A point $t \in \mathbb{T}$ is said to be left dense, left scattered, right dense, right scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. We endow $\mathbb{T}$ with the subspace topology inherited from $\mathbb{R}$.

Now suppose that $x : \mathbb{T} \to \mathbb{R}$. Continuity of x is defined in the usual manner, while x is said to be *ld*-continuous on $\mathbb{T}$ if it is continuous at all left dense points and has finite right sided limits at all right dense points of $\mathbb{T}$. We let $\mathcal{C}_{ld}(\mathbb{T})$ denote the set of *ld*-continuous functions $x : \mathbb{T} \to \mathbb{R}$, and let

$$\|x\| := \max_{t \in \mathbb{T}} |x(t)|, \quad x \in \mathcal{C}_{ld}(\mathbb{T}).$$

With this norm $\mathcal{C}_{ld}$ is a Banach space.

We assume throughout that $\rho^2(b) > \sigma^2(a)$, where $\sigma^2(t) := \sigma(\sigma(t))$ and $\rho^2(t) = \rho(\rho(t))$ so that $\mathbb{T}$ must contained at least 6 points. Now define the sets $\mathbb{T}_\kappa := \mathbb{T} - [a, \sigma(a))$, $\mathbb{T}^\kappa := \mathbb{T} - (\rho(b), b]$, $\mathbb{T}_\kappa^\kappa := \mathbb{T} - ([a, \sigma(a)) \cup (\rho(b), b])$ and $\mathbb{T}_{\kappa^2}^{\kappa^2} := \mathbb{T} - ([a, \sigma^2(a)) \cup (\rho^2(b), b])$. These sets are closed, so they are time-scales and we can also define the above spaces and norms using $\mathbb{T}_\kappa^\kappa$ and $\mathbb{T}_{\kappa^2}^{\kappa^2}$ instead of $\mathbb{T}$.

A function $x : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^\kappa$ if there exists a number $x^\Delta(t)$ with the following property: for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$s \in \mathbb{T} \quad \text{and} \quad |t - s| < \delta \Rightarrow |x(\sigma(t)) - x(s) - x^\Delta(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|.$$

If x is delta differentiable at every $t \in \mathbb{T}^\kappa$ then x is said to be delta differentiable. Similarly, a function $x : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t \in \mathbb{T}_\kappa$ if there exists a number $x^\nabla(t)$ with the following property: for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$s \in \mathbb{T} \quad \text{and} \quad |t - s| < \delta \Rightarrow |x(\rho(t)) - x(s) - x^\nabla(t)(\rho(t) - s)| \le \epsilon|\rho(t) - s|.$$

If x is nabla differentiable at every $t \in \mathbb{T}_\kappa$ then x is said to be nabla differentiable.

A function $F : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\nabla(t) = f(t)$ holds for all $t \in \mathbb{T}_\kappa$. We then define the nabla integral of f by

$$\int_a^t f(s)\nabla s = F(t) - F(a) \quad \text{for all } t \in \mathbb{T}.$$

Every *ld*-continuous function has a nabla antiderivative.

For convenience, we now present some symmetric definitions.

Definition 1.1. A time-scale $\mathbb{T}$ is said to be symmetric if for any given $t \in \mathbb{T}$, we have $b - t + a \in \mathbb{T}$.

Definition 1.2. A function $x : \mathbb{T} \to \mathbb{R}$ is said to be symmetric on $\mathbb{T}$ if for any given $t \in \mathbb{T}$, $x(t) = x(b - t + a)$.

Definition 1.3. We say x is a symmetric solution of FBVP (1.1)–(1.2) on $\mathbb{T}$ provided x is a solution of FBVP (1.1)–(1.2) and is symmetric on $\mathbb{T}$.

Throughout this paper, $\mathbb{T}$ is a symmetric bounded time-scale with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$ and we assume that

(H1) $p \in \mathcal{C}_{ld}(\mathbb{T})$ and p is positive and symmetric on $\mathbb{T}$,

(H2) $f : \mathbb{T} \times [0, \infty) \to [0, \infty)$ is *ld*-continuous, and does not vanish identically, in addition $f(., x)$ is a symmetric function on $\mathbb{T}$, i.e., $f(b - t + a, x) = f(t, x)$ for all $(t, x) \in \mathbb{T} \times [0, \infty)$.

To prove the main results, we will make use of the following lemmas.

Lemma 1.1. *Assume that* (H1) *holds. Let* $y \in \mathcal{C}_{ld}(\mathbb{T})$ *and* $y(t) \not\equiv 0$. *Then the BVP*

$$\phi(-px^{\Delta\nabla})(t) - y(t) = 0, \quad t \in \mathbb{T}_{\kappa}^{\kappa}, \tag{1.3}$$

$$x(a) = x(b) = 0, \tag{1.4}$$

has a unique solution

$$x(t) = \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}(y(s))\nabla s \tag{1.5}$$

where

$$G(t,s) = \frac{1}{b-a}\begin{cases}(t-a)(b-s), & t \le s;\\ (s-a)(b-t), & s \le t.\end{cases} \tag{1.6}$$

Proof. First suppose that $x \in \mathcal{C}_{ld}(\mathbb{T})$ is a solution of (1.3)–(1.4). By integration of both sides of (1.3) from a to t, we have

$$x^{\Delta}(t) = x^{\Delta}(a) - \int_a^t \frac{1}{p(s)}\phi^{-1}(y(s))\nabla s.$$

Integrating again, we have

$$x(t) = x(a) + x^{\Delta}(a)(t-a) - \int_a^t (t-s)\frac{1}{p(s)}\phi^{-1}(y(s))\nabla s. \tag{1.7}$$

Letting $t = b$ in (1.7) and using boundary conditions (1.4), we find

$$x(b) = x^{\Delta}(a)(b-a) - \int_a^b (b-s)\frac{1}{p(s)}\phi^{-1}(y(s))\nabla s = 0$$

then

$$x^{\Delta}(a) = \frac{1}{b-a}\int_a^b (b-s)\frac{1}{p(s)}\phi^{-1}(y(s))\nabla s. \tag{1.8}$$

Substituting (1.8) to (1.7), we have

$$\begin{aligned} x(t) &= \frac{t-a}{b-a}\int_a^b (b-s)\frac{1}{p(s)}\phi^{-1}(y(s))\nabla s - \int_a^t (t-s)\frac{1}{p(s)}\phi^{-1}(y(s))\nabla s \\ &= \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}(y(s))\nabla s \end{aligned} \tag{1.9}$$

where $G(t,s)$ is defined in (1.6).

Sufficiency, let x be as in (1.7), then taking delta differential of (1.7), we have

$$x^{\Delta}(t) = \frac{1}{b-a}\int_a^b (b-s)\frac{1}{p(s)}\phi^{-1}(y(s))\nabla s - \int_a^t \frac{1}{p(s)}\phi^{-1}(y(s))\nabla s$$

and after that taking nabla differential, we have

$$x^{\Delta\nabla}(t) = -\frac{1}{p(t)}\phi^{-1}(y(t)).$$

Furthermore, taking $t = a$ and $t = b$, respectively in (1.9), we are able to obtain the boundary value equation of (1.4). This proof is completed. $\square$

Lemma 1.2. *Assume that* (H2) *holds. Then for* $x \in \mathcal{C}_{ld}(\mathbb{T})$, *the BVP*

$$-y^{\Delta\nabla}(t) = f(t,x(t)), \quad t \in \mathbb{T}_{\kappa^2}^{\kappa^2}, \tag{1.10}$$

$$y(\sigma(a)) = y(\rho(b)) = 0, \tag{1.11}$$

has a unique solution

$$y(t) = \int_{\sigma(a)}^{\rho(b)} H(t,s)f(t,x(t))\nabla s \tag{1.12}$$

where

$$H(t,s)=\frac{1}{\rho(b)-\sigma(a)}\begin{cases}(t-\sigma(a))(\rho(b)-s), & t\le s;\\ (s-\sigma(a))(\rho(b)-t), & s\le t.\end{cases} \tag{1.13}$$

Proof. It can be proved in a way similar to Lemma 1.1.

Assume that x is a solution of problem (1.1)–(1.2). From Lemma 1.1, we have

$$x(t)=\int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}(y(s))\nabla s$$

and then from Lemma 1.2, we have

$$x(t)=\int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s. \quad \square$$

Lemma 1.3. *For $t,s\in\mathbb{T}$, we have $G(t,s)\ge 0$ and $G(t,s)\le G(s,s)$.*

Proof. It is obvious from (1.6). $\square$

Lemma 1.4. *For $t,s\in T_\kappa^\kappa$, we have $H(t,s)\ge 0$ and $H(t,s)\le H(s,s)$.*

Proof. It is obvious from (1.13). $\square$

Lemma 1.5. *Let $\delta\in\left(0,\frac{1}{2}\right)$ be a given constant, then*

$$G(t,s)\ge\frac{\delta}{b-a}G(s,s),\quad t\in[q_1,q_2]\cap\mathbb{T},\ s\in\mathbb{T} \tag{1.14}$$

where $q_1:=\min\{t\in\mathbb{T}: a+\delta\le t\}$ and $q_2:=\max\{t\in\mathbb{T}: t\le b-\delta\}$.

Proof. For $t\le s$, we get

$$\frac{G(t,s)}{G(s,s)}=\frac{t-a}{s-a}\ge\frac{q_1-a}{b-a}\ge\frac{a+\delta-a}{b-a}=\frac{\delta}{b-a}.$$

For $s\le t$, we get

$$\frac{G(t,s)}{G(s,s)}=\frac{b-t}{b-s}\ge\frac{b-q_2}{b-a}\ge\frac{b-b+\delta}{b-a}=\frac{\delta}{b-a}.$$

Therefore, we have (1.14). $\square$

Lemma 1.6. *Let $\delta\in\left(0,\frac{1}{2}\right)$ be a given constant, then we have*

$$H(t,s)\ge\frac{\delta}{\rho(b)-\sigma(a)}H(s,s),\quad t\in[q_1^*,q_2^*]\cap\mathbb{T},\ s\in\mathbb{T}_\kappa^\kappa \tag{1.15}$$

where $q_1^*:=\min\{t\in\mathbb{T}:\sigma(a)+\delta\le t\}$ and $q_2^*:=\max\{t\in\mathbb{T}: t\le\rho(b)-\delta\}$.

Proof. It can be proved in a way similar to Lemma 1.5. $\square$

Lemma 1.7. *Let $\mathbb{T}$ be a bounded symmetric time-scale such that $a=\min\mathbb{T}$, $b=\max\mathbb{T}$.*

Then we have $b-\sigma(a)+a=\rho(b)$ and $b-\rho(b)+a=\sigma(a)$.

Proof. Since $\mathbb{T}$ is a symmetric time-scale, it is obvious. $\square$

Lemma 1.8. *For $t,s\in\mathbb{T}$, we have $G(b-t+a,b-s+a)=G(t,s)$.*

Proof. For $t,s\in\mathbb{T}$ by using (1.6), we have

$$\begin{aligned}G(b-t+a,b-s+a)&=\frac{1}{b-a}\begin{cases}(b-t+a-a)(b-b+s-a), & s\le t;\\ (b-s+a-a)(b-b+t-a), & t\le s\end{cases}\\ &=\frac{1}{b-a}\begin{cases}(s-a)(b-t), & s\le t;\\ (t-a)(b-s), & t\le s\end{cases}\\ &=G(t,s). \quad \square\end{aligned}$$

Lemma 1.9. *For $t,s\in\mathbb{T}_\kappa^\kappa$, we have $H(b-t+a,b-s+a)=H(t,s)$.*

Proof. By using Lemma 1.7 and (1.13), it can be proved in a way similar to Lemma 1.8.

Now, we let $B = \mathcal{C}_{ld}(\mathbb{T})$ then B is a Banach space with $\|x\| = \max_{t\in\mathbb{T}} |x(t)|$, and define a cone $P \subset B$ by

$$P = \{x \in B : x(t) \geq 0,\ x^{\Delta\nabla}(t) \leq 0,\ x(t) \text{ is symmetric on } \mathbb{T} \text{ and } x(t) \geq \gamma \|x\|\}$$

where $\gamma := \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)$.

Second, we define the integral operator $T : P \to B$ by

$$Tx(t) = \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s. \tag{1.16}$$

So, we have

$$\begin{aligned} T^{\Delta}x(t) &= \frac{1}{b-a}\int_a^t (a-s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\ &\quad + \frac{1}{b-a}\int_t^b (b-s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s, \\ T^{\Delta\nabla}x(t) &= -\frac{1}{p(t)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(t,s)f(s,x(s))\nabla s\right). \end{aligned}$$

Hence, for $x \in P$, $Tx(t) \geq 0$ on $\mathbb{T}$ and $T^{\Delta\nabla}x(t) \leq 0$ on $\mathbb{T}_{\kappa}^{\kappa}$.

Using that $p(t)$, $x(t)$ and $f(t, x(t))$ are symmetric on $\mathbb{T}$, we have

$$\begin{aligned} Tx(b-t+a) &= \int_a^b G(b-t+a,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(b-s+a,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\ &= \int_b^a G(b-t+a,b-s+a)\frac{1}{p(b-s+a)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(b-s+a,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla(b-s+a) \\ &= \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(b-s+a,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\ &= \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\rho(b)}^{\sigma(a)} H(b-s+a,b-\tau+a)f(b-\tau+a,x(b-\tau+a))\nabla(b-\tau+a)\right)\nabla s \\ &= \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\ &= Tx(t) \end{aligned}$$

for every $t \in \mathbb{T}$. This implies that $Tx(t)$ is symmetric on $\mathbb{T}$. It is easy to verify that $Tx(t) \geq \gamma\|Tx\|$. So, $T : P \to P$. □

Lemma 1.10. *Assume* (H1) *and* (H2) *hold. Then $x \in B$ is a solution of FBVP (1.1)–(1.2) if and only if x is a fixed point of the operator T.*

Lemma 1.11. *Assume* (H1) *and* (H2) *hold. Then, the operator $T : P \to P$ is completely continuous.*

Proof. Suppose that $K \subset P$ is a bounded set. Let $M > 0$ be such that $\|x\| \leq M$ for $x \in K$, we have

$$\begin{aligned} |Tx(t)| &= \left|\int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\right| \\ &\leq \int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\ &\leq \left\{\int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\right\}\phi^{-1}\left(\sup_{x\in K,\ t\in\mathbb{T}} f(t,x(t))\right) \end{aligned}$$

for every $t \in \mathbb{T}$. This implies that $T(K)$ is bounded. By the Arzela–Ascoli theorem and the Lebesgue dominated convergent theorem on time-scales, we can easily seen that T is s completely continuous operator. □

2. Main result

In this section, we consider the existence of one or two positive symmetric solutions of the FBVP (1.1)–(1.2). Let us define

$$\underline{f_0} = \lim_{x\to 0^+} \min_{t\in[q_1^*,q_2^*]} \frac{f(t,x)}{\phi(x)}, \qquad \overline{f_0} = \overline{\lim_{x\to 0^+}} \max_{t\in\mathbb{T}} \frac{f(t,x)}{\phi(x)}$$
$$\underline{f_\infty} = \underline{\lim_{x\to \infty}} \min_{t\in[q_1^*,q_2^*]} \frac{f(t,x)}{\phi(x)}, \qquad \overline{f_\infty} = \overline{\lim_{x\to \infty}} \max_{t\in\mathbb{T}} \frac{f(t,x)}{\phi(x)}.$$

To prove the results, we will use the following theorem which can be found in Krasnosel'skii's book [18] and in Deimling's book [19].

Theorem 2.1 (*Guo–Krasnosel'skii Fixed Point Theorem*)**.** *Let B be a Banach space, $P \subset B$ be a cone, and suppose that Ω_1 and Ω_2 are open, bounded subsets of P with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|$ *for* $u \in P \cap \partial\Omega_1$, $\|Tu\| \ge \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *or*
(ii) $\|Pu\| \ge \|u\|$ *for* $u \in P \cap \partial\Omega_1$, $\|Pu\| \le \|u\|$ *for* $u \in P \cap \partial\Omega_2$

holds. Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

For convenience, we denote

$$m := \int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\nabla\tau\right)\nabla s$$
$$M := \int_{q_1^*}^{q_2^*} G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(\tau,\tau)\nabla\tau\right)\nabla s.$$

Theorem 2.2. *Assume that* (H1) *and* (H2) *are satisfied. If either $\overline{f_0} = 0, \underline{f_\infty} = \infty$ or $\underline{f_0} = \infty, \overline{f_\infty} = 0$ holds, then the FBVP* (1.1)–(1.2) *has a symmetric positive solution.*

Proof. At first, in view of $\overline{f_0} = \overline{\lim}_{x\to 0^+} \max_{t\in\mathbb{T}} \frac{f(t,x)}{\phi(x)} = 0$ uniformly on $\mathbb{T}$, we may choose an $r_1 > 0$ such that

$$f(t,x) \le \eta\phi(x), \quad 0 \le x \le r_1,\ t \in \mathbb{T},$$

where $\eta < \phi\left(\frac{1}{m}\right)$. Then if Ω_1 is the ball in B centered at the origin with radius r_1 and if $x \in P \cap \partial\Omega_1$, then we have

$$\begin{aligned}
\|Tx\| &= \max_{t\in\mathbb{T}} \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\
&\le \int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\eta\phi(x(\tau))\nabla\tau\right)\nabla s\\
&\le \int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\eta\phi(r_1)\nabla\tau\right)\nabla s\\
&\le \phi^{-1}(\eta)r_1\int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\nabla\tau\right)\nabla s\\
&\le \phi^{-1}(\eta)r_1 m = r_1 = \|x\|,
\end{aligned}$$

and so $\|Tx\| \le \|x\|$ for all $x \in P \cap \partial\Omega_1$.

Next we use the assumption $\underline{f_\infty} = \underline{\lim_{x\to\infty}} \min_{t\in[q_1^*,q_2^*]} \frac{f(t,x)}{\phi(x)} = \infty$ uniformly on $[q_1^*, q_2^*]$. There exists an $r_2 > 0$ large enough such that $f(t,x(t)) \ge \mu\phi(x)$ for $t \in [q_1^*, q_2^*]$, $x \ge r_2$ where $\mu \ge \phi([\frac{\delta}{b-a}\phi^{-1}(\frac{\delta}{\rho(b)-\sigma(a)})M]^{-1})$. If we define $\Omega_2 = \{x \in B : \|x\| < r_2\}$, for $t \in [q_1^*, q_2^*]$ and $x \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
Tx(t_0) &= \int_a^b G(t_0,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\
&\ge \int_{q_1^*}^{q_2^*} G(t_0,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\
&\ge \int_{q_1^*}^{q_2^*} \frac{\delta}{b-a}G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} \frac{\delta}{\rho(b)-\sigma(a)}H(\tau,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)\int_{q_1^*}^{q_2^*} G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(\tau,\tau)\mu\phi(x(\tau))\nabla\tau\right)\nabla s \\
&\geq \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)\phi^{-1}(\mu)r_2\int_{q_1^*}^{q_2^*} G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(\tau,\tau)\nabla\tau\right)\nabla s \\
&= \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)\phi^{-1}(\mu)r_2 M \geq r_2 = \|x\|
\end{aligned}
$$

and so $\|Tx\| \geq \|x\|$ for all $x \in P \cap \partial\Omega_2$. Consequently, by Part (i) of Theorem 2.1, it follows that T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and this implies that our given FBVP (1.1)–(1.2) has a symmetric positive solution.

Next, let $\underline{f_0} = \infty, \overline{f_\infty} = 0$. hold. In view of $\underline{f_0} = \underline{\lim}_{x\to 0^+} \min_{t\in[q_1^*,q_2^*]} \frac{f(t,x)}{\phi(x)} = \infty$, there exists $\overline{r_1} > 0$ such that

$$f(t,x) \geq \overline{\eta}\phi(x),$$

for $t \in [q_1^*, q_2^*]$, $0 < x \leq \bar{r}_1$, such that $\overline{\eta} \geq \mu$ where μ is given in the first part of the proof. Then for $x \in P$ and $\|x\| = \bar{r}_1$, for $t_0 \in [q_1^*, q_2^*]$, we have

$$
\begin{aligned}
Tx(t_0) &= \int_a^b G(t_0,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\
&\geq \int_{q_1^*}^{q_2^*} G(t_0,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\
&\geq \int_{q_1^*}^{q_2^*} \frac{\delta}{b-a}G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} \frac{\delta}{\rho(b)-\sigma(a)}H(\tau,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\
&\geq \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)\int_{q_1^*}^{q_2^*} G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(\tau,\tau)\mu\phi(x(\tau))\nabla\tau\right)\nabla s \\
&\geq \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)\phi^{-1}(\mu)\bar{r}_1\int_{q_1^*}^{q_2^*} G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(\tau,\tau)\nabla\tau\right)\nabla s \\
&= \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)\phi^{-1}(\mu)\bar{r}_1 M \geq \bar{r}_1 = \|x\|.
\end{aligned}
$$

Therefore, if $\Omega_1 \subset B$ is a ball of radius $\bar{r}_1$ centered at the origin, then for $x \in P \cap \partial\Omega_1$, we have $\|Tx\| \geq \|x\|$.

Next, since $\overline{f_\infty} = \overline{\lim}_{x\to\infty} \max_{t\in\mathbb{T}} \frac{f(t,x)}{\phi(x)} = 0$, there exists a $\bar{r}_2 > 0$ such that

$$f(t,x) \leq \eta\phi(x) \quad \text{for } x \geq \bar{r}_2,\ t \in \mathbb{T} \tag{2.17}$$

where $\eta < \phi\left(\frac{1}{m}\right)$. We consider two cases.

Case I. Suppose $f(t,x)$ is bounded on $\mathbb{T}\times[0,\infty)$. In this case, there is an $N > 0$ such that $f(t,x) \leq N$, for $t \in \mathbb{T}$, $x \in [0,\infty)$. In this case, choose $r_2^* \geq \max\left\{2\bar{r}_1, \phi^{-1}(N)m\right\}$. Then for $x \in P$ with $\|x\| = r_2^*$, we have for all $t \in \mathbb{T}$,

$$
\begin{aligned}
\|Tx\| &= \max_{t\in\mathbb{T}} \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\
&\leq \phi^{-1}(N)\int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\nabla\tau\right)\nabla s \\
&\leq \phi^{-1}(N)m \leq r_2^* = \|x\|
\end{aligned}
$$

so that $\|Tx\| \leq \|x\|$.

Case II. Assume $f(t,x)$ is unbounded on $\mathbb{T}\times[0,\infty)$. In this case

$$g(r) := \max\{f(t,x) : t \in \mathbb{T},\ 0 \leq \phi^{-1}(x) \leq r\} \tag{2.18}$$

such that $\lim_{r\to\infty} g(r) = \infty$. Therefore, we can choose $r_2^* > \max\{2r, \bar{r}_2\}$ such that $g(r_2^*) \geq g(r)$ for $0 \leq r \leq r_2^*$. Since $\bar{r}_2 \leq r_2^*$, then from (2.17) and (2.18), we get

$$f(t,x) \leq g(r_2^*) \leq \eta\phi(r_2^*) \quad \text{for } t \in \mathbb{T},\ x \in [0, r_2^*]$$

and hence, for $x \in P$ and $\|x\| = r_2$, we have

$$\begin{aligned}\|Tx\| &= \max_{t\in\mathbb{T}} \int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\ &\le \int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\eta\phi(r_2^*)\nabla\tau\right)\nabla s\\ &\le \phi^{-1}(\eta)r_2^*\int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\nabla\tau\right)\nabla s\\ &= \phi^{-1}(\eta)r_2^* m \le r_2^* m = \|x\|,\end{aligned}$$

and again we have $\|Tx\| \le \|x\|$ for $x \in P \cap \partial\Omega_2$, where $\Omega_2 = \{x \in B : \|x\| \le r_2^*\}$ in both cases. It follows from part (ii) of Theorem 2.1 that T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and this implies that our given FBVP (1.1)–(1.2) has a positive solution. □

Now we will give the sufficient conditions to have two symmetric positive solutions for FBVP (1.1)–(1.2).

(H3) There exists a constant $R_1 > 0$ such that $f(t,x(t)) \le \phi(\frac{R_1}{m})$, for $x \in [0,R_1]$, $t \in \mathbb{T}$.

(H4) There exists a constant $R_2 > 0$ such that $f(t,x(t)) \ge \phi(\frac{R_2}{M\gamma})$, for $x \in [\gamma R_2, R_2]$, $t \in [q_1^*, q_2^*]$.

Theorem 2.3. *Assume that* (H1), (H2) *and* (H3) *are satisfied. If* $\underline{f_0} = \underline{f_\infty} = \infty$ *then the FBVP* (1.1)–(1.2) *has two symmetric positive solutions* x_1 *and* x_2 *such that* $0 < \|x_1\| < R_1 < \|x_2\|$.

Proof. At first, in view of $\underline{f_0} = \underline{\lim}_{x\to 0^+} \min_{t\in[q_1^*,q_2^*]} \frac{f(t,x)}{\phi(x)} = \infty$, there exists an $R^* > 0$ such that

$$f(t,x) \ge \mu\phi(x),$$

for $t \in [q_1^*, q_2^*]$, $0 < x \le R^*$, where μ is chosen so that $\mu \ge \phi(\frac{1}{\gamma^2 M})$. Set $\Omega_1 = \{x \in B : \|x\| < R^*\}$. Then for $x \in P \cap \partial\Omega_1$ and $t_0 \in [q_1^*, q_2^*]$, we have

$$\begin{aligned}Tx(t_0) &= \int_a^b G(t_0,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\ &\ge \int_{q_1}^{q_2} G(t_0,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\ &\ge \int_{q_1}^{q_2} \frac{\delta}{b-a}G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} \frac{\delta}{\rho(b)-\sigma(a)}H(\tau,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\ &\ge \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)\int_{q_1^*}^{q_2^*} G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(\tau,\tau)\mu\phi(x(\tau))\nabla\tau\right)\nabla s\\ &\ge \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)\phi^{-1}(\mu)\gamma\|x\|\int_{q_1^*}^{q_2^*} G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(\tau,\tau)\nabla\tau\right)\nabla s\\ &= \gamma^2\phi^{-1}(\mu)M\|x\| \ge \|x\|\end{aligned}$$

which implies

$$\|Tx\| \ge \|x\| \quad \text{for } x \in P \cap \partial\Omega_1. \tag{2.19}$$

Next, since $\underline{f_\infty} = \underline{\lim}_{x\to\infty} \min_{t\in[q_1^*,q_2^*]} \frac{f(t,x)}{\phi(x)} = \infty$, then for any $\eta \ge \phi(\frac{1}{\gamma^2 M})$, there exists an $R_* > R_1$ such that $f(t,x) \ge \eta\phi(x)$ for $x \ge R_*$. Set $\Omega_2 = \{x \in B : \|x\| < R_*\}$. For $x \in P \cap \partial\Omega_2$ and $t_0 \in [q_1^*, q_2^*]$, since $x \in P$, $x(t) \ge \gamma\|x\| = \gamma R_*$, we have

$$\begin{aligned}Tx(t_0) &= \int_a^b G(t_0,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\ &\ge \int_{q_1}^{q_2} G(t_0,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\\ &\ge \int_{q_1}^{q_2} \frac{\delta}{b-a}G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} \frac{\delta}{\rho(b)-\sigma(a)}H(\tau,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)\int_{q_1^*}^{q_2^*} G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(\tau,\tau)\eta\phi(x(\tau))\nabla\tau\right)\nabla s \\
&\geq \frac{\delta}{b-a}\phi^{-1}\left(\frac{\delta}{\rho(b)-\sigma(a)}\right)\phi^{-1}(\eta)\gamma\|x\|\int_{q_1^*}^{q_2^*} G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{q_1^*}^{q_2^*} H(\tau,\tau)\nabla\tau\right)\nabla s \\
&= \gamma^2\phi^{-1}(\eta)M\|x\| = \|x\|
\end{aligned}
$$

which implies

$$\|Tx\| \geq \|x\| \quad \text{for } x \in P\cap\partial\Omega_2. \tag{2.20}$$

Finally, let $\Omega_3 = \{x \in B : \|x\| < R_1\}$. Then for $x \in P\cap\partial\Omega_3$, from (H3), we have

$$
\begin{aligned}
\|Tx\| &= \max_{t\in\mathbb{T}}\int_a^b G(t,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(s,\tau)f(\tau,x(\tau))\nabla\tau\right)\nabla s \\
&\leq \int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\phi\left(\frac{R_1}{m}\right)\nabla\tau\right)\nabla s \\
&= \frac{R_1}{m}\int_a^b G(s,s)\frac{1}{p(s)}\phi^{-1}\left(\int_{\sigma(a)}^{\rho(b)} H(\tau,\tau)\nabla\tau\right)\nabla s \\
&= R_1 = \|x\|
\end{aligned}
$$

which implies

$$\|Tx\| \leq \|x\| \quad \text{for } x \in P\cap\partial\Omega_3. \tag{2.21}$$

Since $R_* < R_1 < R^*$ and from (2.19)–(2.21), it follows from Theorem 2.1 that T has a fixed point x_1 in $P\cap(\overline{\Omega_3}\setminus\Omega_1)$ and a fixed point x_2 in $P\cap(\overline{\Omega_2}\setminus\Omega_3)$. Both are symmetric positive solutions of the FBVP (1.1)–(1.2) and $0 < \|x_1\| < R_1 < \|x_2\|$. The proof is therefore complete. □

Theorem 2.4. *Assume that* (H1), (H2) *and* (H4) *are satisfied. If* $\overline{f_0} = \overline{f_\infty} = 0$ *then the FBVP* (1.1)–(1.2) *has two symmetric positive solutions* x_1 *and* x_2 *such that* $0 < \|x_1\| < R_2 < \|x_2\|$.

Proof. It can be proved in a way similar to second part of Theorems 2.2 and 2.3. □

Example 2.1. Let $\mathbb{T} = [1, \frac{4}{3}] \cup [\frac{5}{3}, 2]$ be a bounded symmetric time-scale. We consider the following problem:

$$\phi(-t^2(3-t)^2x^{\Delta\nabla})^{\Delta\nabla}(t) + f(t,x(t)) = 0, \quad t \in \mathbb{T}_{\kappa^2}^{\kappa^2}, \tag{2.22}$$

$$x(1) = x(2) = 0, \qquad x^{\Delta\nabla}(1) = x^{\Delta\nabla}(2) = 0 \tag{2.23}$$

where $\phi(x) = \begin{cases} \frac{x^7}{1+x^2}, & x \leq 0; \\ x^2, & x > 0, \end{cases}$ and $f(t,x(t)) = \begin{cases} t(3-t)x^4(t), & (t,x) \in [1,2]\times(0,4]; \\ t(3-t)4x^3(t), & (t,x)\in[1,2]\times[4,\infty). \end{cases}$.

We notice that $a = 1$, $b = 2$, $\sigma(a) = 1$, $\rho(b) = 2$ and $p(t) = t^2(3-t)^2$ is symmetric.

Let $\delta = \frac{1}{3} \in (0, \frac{1}{2})$, then $q_1^* = \min\{t \in \mathbb{T} : 1+\delta \leq t\} = \frac{4}{3}$ and $q_2^* = \max\{t \in \mathbb{T} : t \leq 2-\delta\} = \frac{5}{3}$ we have

$$\overline{f_0} = \overline{\lim_{x\to 0^+}}\max_{t\in\mathbb{T}}\frac{f(t,x)}{\phi(x)} = \overline{\lim_{x\to 0^+}}\max_{t\in\mathbb{T}}\frac{t(3-t)x^4}{x^2} = \lim_{x\to 0^+}\frac{3}{2}(3-\frac{3}{2})x^2 = 0$$

$$\underline{f_\infty} = \underline{\lim_{x\to\infty}}\min_{t\in\{\frac{4}{3},\frac{5}{3}\}}\frac{f(t,x)}{\phi(x)} = \underline{\lim_{x\to\infty}}\min_{t\in\{\frac{4}{3},\frac{5}{3}\}}\frac{t(3-t)4x^3}{x^2} = \lim_{x\to\infty}\frac{20}{9}4x = \infty.$$

Therefore, from Theorem 2.2 the FBVP (2.22)–(2.23) has a symmetric positive solution.

Example 2.2. Let $\mathbb{T} = \{-10, -9, \ldots, -1, 0, 1, \ldots, 9, 10\}$ be a bounded symmetric time-scale. We consider the following problem:

$$\phi\left(\frac{1}{t^2+1}x^{\Delta\nabla}\right)^{\Delta\nabla}(t) + f(t,x(t)) = 0, \quad t \in \mathbb{T}_{\kappa^2}^{\kappa^2}, \tag{2.24}$$

$$x(-10) = x(10) = 0, \qquad x^{\Delta\nabla}(-9) = x^{\Delta\nabla}(9) = 0 \tag{2.25}$$

where $\phi(x) = |x|^{-\frac{1}{2}}x$ and

$$f(t,x(t)) = \begin{cases} (t^2+1)x^{\frac{1}{4}}(t), & (t,x) \in [-10,10] \times (0, 16.10^{16}); \\ (t^2+1)2.10^4 + (x(t) - 16.10^{16})e^{x(t)}, & (t,x) \in [-10,10] \times [16.10^{16}, \infty). \end{cases}$$

We notice that $a = -10, b = 10, \sigma(a) = -9, \rho(b) = 9$ and $p(t) = \frac{1}{t^2+1}$ satisfies the condition $p(-t) = p(t)$ so it is symmetric on $[-10, 10]$.

Choose $\delta = \frac{1}{10} \in (0, \frac{1}{2})$, then $q_1^* = \min\{t \in \mathbb{T} : -9 + \delta \le t\} = -8$ and $q_2^* = \max\{t \in \mathbb{T} : t \le 9 - \delta\} = 8$ and also we have

$$\underline{f_0} = \lim_{x\to 0^+} \min_{t\in[q_1^*,q_2^*]} \frac{(t^2+1)\sqrt[4]{x}}{\sqrt{x}} = \lim_{x\to 0^+} \min_{t\in[-8,8]} \frac{(t^2+1)\sqrt[4]{x}}{\sqrt{x}} = \lim_{x\to 0^+} \frac{\sqrt[4]{x}}{\sqrt{x}} = \infty,$$

$$\begin{aligned} \underline{f_\infty} &= \lim_{x\to\infty} \min_{t\in[q_1^*,q_2^*]} \frac{f(t,x)}{\phi(x)} = \lim_{x\to\infty} \min_{t\in[-8,8]} \frac{(t^2+1)2.10^4 + (x - 16.10^{16})e^x}{\sqrt{x}} \\ &= \lim_{x\to\infty} \frac{2.10^4 + (x-16.10^{16})e^x}{\sqrt{x}} = \infty. \end{aligned}$$

Furthermore we obtain $m \cong 40$ and $M \cong 38$. If we choose $R_1 = 16.10^{16}$, then it is straightforward from Theorem 2.3 that the FBVP (2.24)–(2.25) has two symmetric positive solutions satisfying $0 < \|x_1\| < 16.10^{16} < \|x_2\|$.

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References

[1] S. Hilger, Ein Masskettenkalkül mit Anwendug auf zentrumsmanningfaltigkeiten, Ph.D. Thesis, Universitat Würzburg, 1988.
[2] F.M. Atici, D.C. Biles, A. Lebedinsky, An application of time scales to economics, Math. Comput. Modelling 43 (2006) 718–726.
[3] M.A. Jones, B. Song, D.M. Thomas, Controlling wound healing through debridement, Math. Comput. Modelling 40 (2004) 1057–1064.
[4] V. Spedding, Taming nature's numbers, New Sci. (2003) 28–32.
[5] D.M. Thomas, L. Vandemuelebroeke, K. Yamaguchi, A mathematical evolution model for phytoremediation of metals, Discrete Contin. Dyn. Syst. Ser. B 5 (2005) 411–422.
[6] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhäuser, Boston, 2001.
[7] M. Bohner, A. Peterson (Eds.), Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[8] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.
[9] F.Merdivenci Atici, G.Sh. Guseinov, On Green's functions and positive solutions for boundary value problems on time scales, J. Comput. Appl. Math. 141 (1–2) (2002) 75–99.
[10] F.Merdivenci Atici, S. Gulsan Topal, Nonlinear three point boundary value problems on time scales, Dynam. Systems Appl. 13 (2004) 327–337.
[11] D.R. Anderson, A. Cabada, Third order right-focal multipoint problems on time scales, J. Difference Equ. Appl. 12 (9) (2006) 919–935.
[12] D.R. Anderson, C.C. Tisdell, Third-order nonlocal problems with sign-changing nonlinearity on time scales, Electron. J. Differential Equations 19 (2007) 1–12.
[13] D.R. Anderson, J. Hoffacker, Existence of solutions to a third-order multipoint problem on time scales, Electron. J. Differential Equations 107 (2007) 1–15.
[14] R. Avery, J. Henderson, Existence of three positive pseudo-symmetric solutions for a one dimensional discrete p-Laplacian, J. Difference Equ. Appl. 10 (2004) 529–539.
[15] S.W. Ng, A.D. Rae, The pseudo symmetric structure of bis(dicyclohexylammonium) bis (oxalatotriphenylotannate), Z. Krist. 215 (2000) 199–206.
[16] M. Pei, S.K. Chang, Monotone iterative technique and symmetric positive solutions for a fourth-order boundary value problem, Math. Comput. Modelling 51 (2010) 1260–1267.
[17] Y. Luo, Z. Luo, Symmetric positive solutions for nonlinear boundary value problems with ϕ-Laplacian operator, Appl. Math. Lett. 23 (2010) 657–664.
[18] M. Krasnosel'skii, Positive Solutions of Operator Equations, Noordhoff, Groningen, 1964.
[19] K. Deimling, Nonlinear Functional Analysis, Springer, New York, 1985.